$\mathrm{BC_N}$-graded Lie algebras arising from fermionic representations

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Abstract

We use fermionic representations to obtain a class of $\mathrm{BC_N}$-graded Lie algebras coordinatized by quantum tori with nontrivial central extensions.

0 Introduction

Lie algebras graded by the reduced finite root systems were first introduced by Berman-Moody [BM] in order to understand the generalized intersection matrix algebras of Slodowy. [BM] classified Lie algebras graded by the root systems of type $A_l, l \geq 2$, $D_l, l \geq 4$ and E_6, E_7, E_8 up to central extensions. Benkart-Zelmanov [BZ] classified Lie algebras graded by the root systems of type $A_1, B_l, l \geq 2$, $C_l, l \geq 3$, F_4 and G_2 up to central extensions. Neher [N] gave a different approach for all reduced root systems except E_8, F_4 and G_2. The idea of root graded Lie algebras can be traced back to Tits [T] and Seligman [S]. [ABG1] completed the classification of the above root graded Lie algebras by figuring out explicitly the centers of the universal coverings of those root graded Lie algebras. It turns out that the classification of those root graded Lie algebras played a crucial role in classifying the newly developed extended affine Lie algebras (see [BGKN] and [AG]). All affine Kac-Moody Lie algebras except $A_{2l}^{(2)}$ are examples of Lie algebras graded by reduced finite root systems.

To include the twisted affine Lie algebra $A_{2l}^{(2)}$ and for the purpose of the classification of the extended affine Lie algebras of non-reduced types, [ABG2]

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introduced Lie algebras graded by the non-reduced root system BC_N. BC_N-graded Lie algebras do appear not only in the extended affine Lie algebras (see [AABGP]) including the twisted affine Lie algebra $A_{2l}^{(2)}$ but also in the finite-dimensional isotropic simple Lie algebras studied by Seligman [S]. The other important examples include the "odd symplectic" Lie algebras studied by Gelfand-Zelevinsky [GeZ], Maliakas [Ma] and Proctor [P].

The Cliffold(or Weyl) algebras have natural representations on the exterior(or symmetric) algebras of polynomials over half of generators. Those representations are important in quantum and statistical mechanics where the generators are interpreted as operators which create or annihilate particles and satisfy Fermi(or Bose) statistics. Fermionic representations for the affine Kac-Moody Lie algebras were first developed by Frenkel [F1] and Kac-Peterson [KP] independently. Feingold-Frenkel [FF] systematically constructed representations for all classical affine Lie algebras by using Clifford or Weyl algebras with infinitely many generators. [G] constructed bosonic and fermionic representations for the extended affine Lie algebra $\widetilde{gl_N(\mathbb{C}_q)}$, where $\mathbb{C}_q$ is the quantum torus in two variables. Thereafter Lau [L] gave a more general construction.

In this paper, we will construct fermions depending on the parameter q which will lead to representations for some BC_N-graded Lie algebras coordinatized by quantum tori with nontrivial central extensions. Since C_N-graded Lie algebras are also BC_N-graded Lie algebras we will treat bosons as well in a unified way.

The organization of the paper is as follows. In Section 1, we review the definition of BC_N-graded Lie algebras and give examples of BC_N-graded Lie algebras which are subalgebras of $\widehat{gl_{2N}(\mathbb{C}_q)}$ or $\widehat{gl_{2N+1}(\mathbb{C}_q)}$. In Section 2, we use fermions or bosons to construct representations for those examples of BC_N-graded Lie algebras by using Clifford or Weyl algebras with infinitely many generators. Although we get BC_N-graded Lie algebras with the grading subalgebras of type B_N, C_N and D_N, there is only one which is a genuine BC_N-graded Lie algebra arising from the fermionic construction.

Throughout this paper, we denote the field of complex numbers and the ring of integers by $\mathbb{C}$ and $\mathbb{Z}$ respectively.

1 $\mathrm{BC_N}$-graded Lie Algebras

In this section, we first recall the definition of quantum tori and BC_N-graded Lie algebras. We then go on to provide some examples of BC_N-graded Lie algebras. For more information on BC_N-graded Lie algebras, see [ABG2].

Let q be a non-zero complex number. A quantum torus associated to q (see [M]) is the unital associative $\mathbb{C}$-algebra $\mathbb{C}_q[x^{\pm}, y^{\pm}]$ (or simply $\mathbb{C}_q$) with generators $x^{\pm}$,$y^{\pm}$ and relations

$$xx^{-1} = x^{-1}x = yy^{-1} = y^{-1}y = 1 \quad and \quad yx = qxy. \tag{1.1}$$

Then

$$x^m y^n x^p y^s = q^{np} x^{m+p} y^{n+s} \tag{1.2}$$

and

$$\mathbb{C}_q = \sum_{m,n\in\mathbb{Z}} \oplus \mathbb{C} x^m y^n. \tag{1.3}$$

Set $\Lambda(q) = \{n \in \mathbb{Z} | q^n = 1\}$. From [BGK] we see that $[\mathbb{C}_q, \mathbb{C}_q]$ has a basis consisting of monomials $x^m y^n$ for $m \notin \Lambda(q)$ or $n \notin \Lambda(q)$.

Let $^-$ be the anti-involution on $\mathbb{C}_q$ given by

$$\bar{x} = x, \quad \bar{y} = y^{-1}. \tag{1.4}$$

We have $\mathbb{C}_q = \mathbb{C}_q^+ \oplus \mathbb{C}_q^-$, where $\mathbb{C}_q^{\pm} = \{s \in \mathbb{C}_q | \bar{s} = \pm s\}$, then

$$\begin{aligned} \mathbb{C}_q^+ &= span\{x^m y^n + \overline{x^m y^n} | m \in \mathbb{Z}, n \geq 0\}, \\ \mathbb{C}_q^- &= span\{x^m y^n - \overline{x^m y^n} | m \in \mathbb{Z}, n > 0\}. \end{aligned} \tag{1.5}$$

Now we form a central extension of $gl_r(\mathbb{C}_q)$ (cf. [G]),

$$\widehat{gl_r(\mathbb{C}_q)} = gl_r(\mathbb{C}_q) \oplus \Big(\sum_{n\in\Lambda(q)} \oplus \mathbb{C}c(n) \Big) \oplus \mathbb{C}c_y \tag{1.6}$$

with Lie bracket

$$\begin{aligned} [e_{ij}(x^m y^n), e_{kl}(x^p y^s)] \;=\; & \delta_{jk} q^{np} e_{il}(x^{m+p} y^{n+s}) - \delta_{il} q^{ms} e_{kj}(x^{m+p} y^{n+s}) \\ & + m q^{np} \delta_{jk} \delta_{il} \delta_{m+p,0} \delta_{\overline{n+s},\bar{0}} c(n+s) \\ & + n q^{np} \delta_{jk} \delta_{il} \delta_{m+p,0} \delta_{n+s,0} c_y \end{aligned} \tag{1.7}$$

for $m, p, n, s \in \mathbb{Z}$, where $c(u)$, for $u \in \Lambda(q)$ and c_y are central elements of $\widehat{gl_r(\mathbb{C}_q)}$, $\bar{t}$ means $\bar{t} \in \mathbb{Z}/\Lambda(q)$, for $t \in \mathbb{Z}$.

Next we recall the definition of BC_N-graded Lie algebra and construct three types of BC_N-graded Lie algebras. Let

$$\begin{aligned} \Delta_B &= \{\pm\epsilon_i \pm \epsilon_j | 1 \le i \ne j \le N\} \cup \{\pm\epsilon_i | i = 1, \cdots, N\} \\ \Delta_C &= \{\pm\epsilon_i \pm \epsilon_j | 1 \le i \ne j \le N\} \cup \{\pm 2\epsilon_i | i = 1, \cdots, N\} \\ \Delta_D &= \{\pm\epsilon_i \pm \epsilon_j | 1 \le i \ne j \le N\}. \end{aligned} \tag{1.8}$$

be root systems of type B,C and D respectively, and

$$\Delta = \{\pm\epsilon_i \pm \epsilon_j | 1 \le i \ne j \le N\} \cup \{\pm\epsilon_i, \pm 2\epsilon_i | i = 1, \cdots, N\} \tag{1.9}$$

be a root system of type BC_N in the sense of Bourbaki [Bo, Chapitre VI].

Definition 1.1 ($\mathbf{BC}_N$-graded Lie Algebras) *A Lie algebra L over a field $\mathbb{F}$ of characteristic 0 is graded by the root system BC_N or is BC_N-graded if*

(i) *L contained as a subalgebra a finite-dimentional split "simple" Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\mu \in \Delta_X} \mathfrak{g}_\mu$ whose root system relative to a split Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$ is Δ_X, X=B,C, or D;*

(ii) *$L = \bigoplus_{\mu \in \Delta \cup \{0\}} L_\mu$, where $L_\mu = \{x \in L | [h, x] = \mu(h)x, \text{ for all } h \in \mathfrak{h}\}$ for $\mu \in \Delta \cup \{0\}$, and Δ is the root system BC_N as in (1.9); and*

(iii) *$L_0 = \sum_{\mu \in \Delta} [L_\mu, L_{-\mu}]$.*

In Definition 1.1 the word simple is in quotes, because in every case but two the Lie algebra $\mathfrak{g}$ associated with Δ_{X} is simple; the sole exceptions being when $\Delta_{\mathrm{X}} = \mathrm{D}_2$ or D_1. The D_2 root system is the same as $\mathrm{A}_1 \times \mathrm{A}_1$, and $\mathfrak{g}$ is the sum $\mathfrak{g} = \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)}$ of two copies of $\mathfrak{sl}_2$ in this case. In the D_1 case, $\mathfrak{g} = \mathbb{F}h$, a one-dimensional subalgebra.

We refer to $\mathfrak{g}$ as the grading subalgebra of L, and we say L is BC_N-graded with grading subalgebra $\mathfrak{g}$ of type X_N (where X = B, C, or D) to mean that the root system of $\mathfrak{g}$ is of type X_N.

Any Lie algebra which is graded by a finite root system of type B_N, C_N, or D_N is also BC_N-graded with grading subalgebra of type B_N, C_N, or D_N respectively. For such a Lie algebra L, the space $L_\mu = (0)$ for all μ not in Δ_B, Δ_C, or Δ_D respectively.

1.1 Type C and D

For BC$_N$-graded Lie algebras with grading subalgebra of type $C_N(\rho = -1)$ and $D_N(\rho = 1)$, we put

$$G = \begin{pmatrix} 0 & I_N \\ \rho I_N & 0 \end{pmatrix} \in M_{2N}(\mathbb{C}_q).$$

Then, G is an invertible $2N \times 2N-$matrix and $\bar{G}^t = \rho G$. Using the matrix G, we define a map

$$^* : M_{2N}(\mathbb{C}_q) \to M_{2N}(\mathbb{C}_q) \text{ by } A^* = G^{-1}\bar{A}^t G.$$

Since $\bar{G}^t = \rho G$, * is an involution of the associative algebra $M_{2N}(\mathbb{C}_q)$. As in [AABGP], we define

$$S_\rho(M_{2N}(\mathbb{C}_q),^*) = \{A \in M_{2N}(\mathbb{C}_q) : A^* = -A\}$$

in which case $S_\rho(M_{2N}(\mathbb{C}_q),^*)$ is a Lie subalgebra of $gl_{2N}(\mathbb{C}_q)$ over $\mathbb{C}$. The general form of a matrix in $S_\rho(M_{2N}(\mathbb{C}_q),^*)$ is

$$\begin{pmatrix} A & S \\ T & -\bar{A}^t \end{pmatrix} \quad \text{with } \bar{S}^t = -\rho S \quad \text{and} \quad \bar{T}^t = -\rho T \tag{1.10}$$

where $A, S, T \in M_N(\mathbb{C}_q)$. Then the Lie algebra

$$\mathcal{G}_\rho = [S_\rho(M_{2N}(\mathbb{C}_q),^*), S_\rho(M_{2N}(\mathbb{C}_q),^*)]$$

is a BC$_N$-graded Lie algebra with grading subalgebra of type $C_N(\rho = -1)$ and $D_N(\rho = 1)$. Using the method in [AABGP], we easily know that

$$\mathcal{G}_\rho = \big\{Y \in S_\rho(M_{2N}(\mathbb{C}_q),^*) | tr(Y) \equiv 0 \text{ mod } [\mathbb{C}_q, \mathbb{C}_q]\big\}.$$

We put

$$\mathcal{H} = \Big\{\sum_{i=1}^N a_i(e_{ii} - e_{N+i,N+i}) | a_i \in \mathbb{C}\Big\}, \tag{1.11}$$

then $\mathcal{H}$ is a N-dimensional abelian subalgebra of $\mathcal{G}_\rho$. Defining $\epsilon_i \in \mathcal{H}^*, i = 1, \cdots, N$, by

$$\epsilon_i\Big(\sum_{j=1}^N a_j(e_{jj} - e_{N+j,N+j})\Big) = a_i \tag{1.12}$$

for $i = 1, \cdots, N$. Putting $\mathcal{G}_\alpha = \{x \in \mathcal{G}_\rho | [h, x] = \alpha(h)x, \text{ for all } h \in \mathcal{H}\}$ as usual, we have

$$\mathcal{G}_\rho = \mathcal{G}_0 \oplus \sum_{i \neq j} \mathcal{G}_{\epsilon_i - \epsilon_j} \oplus \sum_{i<j} (\mathcal{G}_{\epsilon_i + \epsilon_j} \oplus \mathcal{G}_{-\epsilon_i - \epsilon_j}) \oplus \sum_i (\mathcal{G}_{2\epsilon_i} \oplus \mathcal{G}_{-2\epsilon_i}) \tag{1.13}$$

where

$$\begin{aligned}
\mathcal{G}_{\epsilon_i - \epsilon_j} &= span_{\mathbb{C}}\{\tilde{f}_{ij}(m, n) = x^m y^n e_{ij} - \overline{x^m y^n} e_{N+j, N+i} | m, n \in \mathbb{Z}\}, \\
\mathcal{G}_{\epsilon_i + \epsilon_j} &= span_{\mathbb{C}}\{\tilde{g}_{ij}(m, n) = x^m y^n e_{i, N+j} - \rho \overline{x^m y^n} e_{j, N+i} | m, n \in \mathbb{Z}\}, \\
\mathcal{G}_{-\epsilon_i - \epsilon_j} &= span_{\mathbb{C}}\{\tilde{h}_{ij}(m, n) = \rho x^m y^n e_{N+i, j} - \overline{x^m y^n} e_{N+j, i} | m, n \in \mathbb{Z}\}, \\
\mathcal{G}_{2\epsilon_i} &= span_{\mathbb{C}}\{\tilde{g}_{ii}(m, n) = (x^m y^n - \rho \overline{x^m y^n}) e_{i, N+i} | m, n \in \mathbb{Z}\}, \\
\mathcal{G}_{-2\epsilon_i} &= span_{\mathbb{C}}\{\tilde{h}_{ii}(m, n) = (\rho x^m y^n - \overline{x^m y^n}) e_{N+i, i} | m, n \in \mathbb{Z}\},
\end{aligned} \tag{1.14}$$

and

$$\mathcal{G}_0 = span_{\mathbb{C}}\{\tilde{f}_{ii}(m, n) - \tilde{f}_{11}(m, n), \tilde{f}_{11}(p, s) | 2 \leq i \leq N, m, n \in \mathbb{Z}, p \notin \Lambda(q) \text{ or } s \notin \Lambda(q)\}.$$

Note that $\tilde{g}_{ij}(m, n) = -\rho q^{-mn} \tilde{g}_{ji}(m, -n)$, $\tilde{h}_{ij}(m, n) = -\rho q^{-mn} \tilde{h}_{ji}(m, -n)$.

Now we form a central extension of $\mathcal{G}_\rho$

$$\widehat{\mathcal{G}}_\rho = \mathcal{G}_\rho \oplus \Big(\sum_{n \in \Lambda(q)} \oplus \mathbb{C}c(n) \Big) \oplus \mathbb{C}c_y \tag{1.15}$$

with Lie brackets as (1.7).

We have

Proposition 1.1

$$[\tilde{g}_{ij}(m, n), \tilde{g}_{kl}(p, s)] = 0 \tag{1.16}$$

(1.17)
$$[\tilde{g}_{ij}(m, n), \tilde{f}_{kl}(p, s)] = -\delta_{il} q^{ms} \tilde{g}_{kj}(m + p, n + s) + \rho \delta_{jl} q^{(s-n)m} \tilde{g}_{ki}(m + p, s - n)$$

$$\begin{aligned}
& [\tilde{g}_{ij}(m, n), \tilde{h}_{kl}(p, s)] \\
= \; & -\delta_{ik} q^{-n(m+p)} \tilde{f}_{jl}(m + p, s - n) + \rho \delta_{jk} q^{np} \tilde{f}_{il}(m + p, n + s) \\
& + \rho \delta_{il} q^{-(mn+np+ps)} \tilde{f}_{jk}(m + p, -(n + s)) - \delta_{jl} q^{(n-s)p} \tilde{f}_{ik}(m + p, n - s) \\
& + m \rho q^{np} \delta_{jk} \delta_{il} \delta_{m+p,0} \delta_{\overline{n+s}, \overline{0}} (c(n + s) + c(-n - s)) \\
& - m \delta_{ik} \delta_{jl} \delta_{m+p,0} \delta_{\overline{n-s}, \overline{0}} (c(n - s) + c(s - n))
\end{aligned} \tag{1.18}$$

$$[\tilde{f}_{ij}(m,n),\tilde{f}_{kl}(p,s)] = \delta_{jk}q^{np}\tilde{f}_{il}(m+p,n+s)-\delta_{il}q^{sm}\tilde{f}_{kj}(m+p,n+s) + 2mq^{np}\delta_{jk}\delta_{il}\delta_{m+p,0}\delta_{\overline{n+s},\overline{0}}c(n+s) \tag{1.19}$$

$$[\tilde{f}_{ij}(m,n),\tilde{h}_{kl}(p,s)] = -\delta_{ik}q^{-n(m+p)}\tilde{h}_{jl}(m+p,s-n)-\delta_{il}q^{ms}\tilde{h}_{kj}(m+p,n+s) \tag{1.20}$$

$$[\tilde{h}_{ij}(m,n),\tilde{h}_{kl}(p,s)] = 0 \tag{1.21}$$

for all $m,p,n,s\in\mathbb{Z}$ *and* $1\le i,j,k,l\le N$.

Proof. We only check (1.18).

$$\begin{aligned}
& [\tilde{g}_{ij}(m,n),\tilde{h}_{kl}(p,s)]\\
= & [x^my^ne_{i,N+j}-\rho\overline{x^my^n}e_{j,N+i},\rho x^py^se_{N+k,l}-\overline{x^py^s}e_{N+l,k}]\\
= & \rho[x^my^ne_{i,N+j},x^py^se_{N+k,l}]-[x^my^ne_{i,N+j},\overline{x^py^s}e_{N+l,k}]-[\overline{x^my^n}e_{j,N+i},x^py^se_{N+k,l}]\\
& +\rho[\overline{x^my^n}e_{j,N+i},\overline{x^py^s}e_{N+l,k}]\\
= & \rho\big(\delta_{jk}x^my^nx^py^se_{il}-\delta_{il}x^py^sx^my^ne_{N+k,N+j}+mq^{np}\delta_{jk}\delta_{il}\delta_{m+p,0}\delta_{\overline{n+s},\overline{0}}c(n+s)\big)\\
& -\big(\delta_{jl}x^my^n\overline{x^py^s}e_{ik}-\delta_{ki}\overline{x^py^s}x^my^ne_{N+l,N+j}+m\delta_{jl}\delta_{ik}\delta_{m+p,0}\delta_{\overline{n-s},\overline{0}}c(n-s)\big)\\
& -\big(\delta_{ik}\overline{x^my^n}x^py^se_{jl}-\delta_{lj}x^py^s\overline{x^my^n}e_{N+k,N+i}+m\delta_{jl}\delta_{ik}\delta_{m+p,0}\delta_{\overline{n-s},\overline{0}}c(s-n)\big)\\
& +\rho\big(\delta_{il}\overline{x^py^sx^my^n}e_jk-\delta_{kj}\overline{x^my^nx^py^s}e_{N+l,N+i}+mq^{np}\delta_{jl}\delta_{ik}\delta_{m+p,0}\delta_{\overline{n+s},\overline{0}}c(-n-s)\big)\\
& +\rho n\delta_{jk}\delta_{il}\delta_{m+p,0}\delta_{n+s,0}c_y-n\delta_{jl}\delta_{ik}\delta_{m+p,0}\delta_{n-s,0}c_y+n\delta_{jl}\delta_{ik}\delta_{m+p,0}\delta_{n-s,0}c_y\\
& -\rho n\delta_{jk}\delta_{il}\delta_{m+p,0}\delta_{n+s,0}c_y\\
= & -\delta_{ik}q^{-n(m+p)}\tilde{f}_{jl}(m+p,s-n)+\rho\delta_{jk}q^{np}\tilde{f}_{il}(m+p,n+s)\\
& +\rho\delta_{il}q^{-(mn+np+ps)}\tilde{f}_{jk}(m+p,-(n+s))-\delta_{jl}q^{(n-s)p}\tilde{f}_{ik}(m+p,n-s)\\
& +m\rho q^{np}\delta_{jk}\delta_{il}\delta_{m+p,0}\delta_{\overline{n+s},\overline{0}}(c(n+s)+c(-n-s))\\
& -m\delta_{ik}\delta_{jl}\delta_{m+p,0}\delta_{\overline{n-s},\overline{0}}(c(n-s)+c(s-n)).
\end{aligned}$$

The proof of the others is similar. ■

1.2 Type B

For type B, we put

$$G=\begin{pmatrix}1&0&0\\0&0&I_N\\0&I_N&0\end{pmatrix}\in M_{2N+1}(\mathbb{C}_q).$$

Then, G is an invertible $(2N+1)\times(2N+1)$-matrix and $\bar{G}^t = G$. Using the matrix G, we define a map

$$^*: M_{2N+1}(\mathbb{C}_q) \to M_{2N+1}(\mathbb{C}_q) \text{ by } A^* = G^{-1}\bar{A}^t G.$$

Since $\bar{G}^t = G$, * is an involution of the associative algebra $M_{2N+1}(\mathbb{C}_q)$. As in [AABGP], we define

$$S(M_{2N+1}(\mathbb{C}_q),^*) = \{A \in M_{2N+1}(\mathbb{C}_q) : A^* = -A\}$$

in which case $S(M_{2N+1}(\mathbb{C}_q),^*)$ is a Lie subalgebra of $gl_{2N+1}(\mathbb{C}_q)$ over $\mathbb{C}$. The general form of a matrix in $S(M_{2N+1}(\mathbb{C}_q),^*)$ is

$$(1.22)\qquad \begin{pmatrix} a & b_1 & b_2 \\ -\bar{b_2}^t & A & S \\ -\bar{b_1}^t & T & -\bar{A}^t \end{pmatrix} \quad \text{with } \bar{a} = -a \quad \bar{S}^t = -S \quad \text{and} \quad \bar{T}^t = -T$$

where $A, S, T \in M_N(\mathbb{C}_q)$. Then the Lie algebra

$$\mathcal{G}' = [S(M_{2N+1}(\mathbb{C}_q),^*), S(M_{2N+1}(\mathbb{C}_q),^*)]$$

is a BC_N-graded Lie algebra with grading subalgebra of type B_N. Following from [AABGP], we easily know that

$$\mathcal{G}' = \{Y \in S(M_{2N+1}(\mathbb{C}_q),^*) | tr(Y) \equiv 0 \bmod [\mathbb{C}_q, \mathbb{C}_q]\}$$

As in Section 1.1, we set

$$(1.23)\qquad \mathcal{H}' = \Big\{\sum_{i=1}^{N} a_i(e_{ii} - e_{N+i,N+i}) | a_i \in \mathbb{C}\Big\},$$

then $\mathcal{H}'$ is a N-dimensional abelian subalgebra of $\mathcal{G}'$. Defining $\epsilon_i \in \mathcal{H}'^*, i = 1, \cdots, N$, by

$$(1.24)\qquad \epsilon_i\Big(\sum_{j=1}^{l} a_j(e_{jj} - e_{N+j,N+j})\Big) = a_i$$

for $i = 1, \cdots, N$. Putting $\mathcal{G}'_\alpha = \{x \in \mathcal{G}' | [h, x] = \alpha(h)x, \text{ for all } h \in \mathcal{H}'\}$ as usual, we have

$$(1.25)\quad \mathcal{G}' = \mathcal{G}'_0 \oplus \sum_{i\neq j} \mathcal{G}'_{\epsilon_i - \epsilon_j} \oplus \sum_{i<j} (\mathcal{G}'_{\epsilon_i+\epsilon_j} \oplus \mathcal{G}'_{-\epsilon_i-\epsilon_j}) \oplus \sum_i (\mathcal{G}'_{\epsilon_i} \oplus \mathcal{G}'_{-\epsilon_i} \oplus \mathcal{G}'_{2\epsilon_i} \oplus \mathcal{G}'_{-2\epsilon_i})$$

where

$$
\begin{aligned}
\mathcal{G}'_{\epsilon_i-\epsilon_j} &= span_{\mathbb{C}}\{\tilde{f}_{ij}(m,n) = x^m y^n e_{ij} - \overline{x^m y^n} e_{N+j,N+i} | m,n \in \mathbb{Z}\}, \\
\mathcal{G}'_{\epsilon_i+\epsilon_j} &= span_{\mathbb{C}}\{\tilde{g}_{ij}(m,n) = x^m y^n e_{i,N+j} - \overline{x^m y^n} e_{j,N+i} | m,n \in \mathbb{Z}\}, \\
\mathcal{G}'_{-\epsilon_i-\epsilon_j} &= span_{\mathbb{C}}\{\tilde{h}_{ij}(m,n) = x^m y^n e_{N+i,j} - \overline{x^m y^n} e_{N+j,i} | m,n \in \mathbb{Z}\}, \\
\mathcal{G}'_{2\epsilon_i} &= span_{\mathbb{C}}\{\tilde{g}_{ii}(m,n) = (x^m y^n - \overline{x^m y^n}) e_{i,N+i} | m,n \in \mathbb{Z}\}, \\
\mathcal{G}'_{-2\epsilon_i} &= span_{\mathbb{C}}\{\tilde{h}_{ii}(m,n) = (x^m y^n - \overline{x^m y^n}) e_{N+i,i} | m,n \in \mathbb{Z}\}, \\
\mathcal{G}'_{\epsilon_i} &= span_{\mathbb{C}}\{\tilde{e}_{i}(m,n) = x^m y^n e_{i,0} - \overline{x^m y^n} e_{0,N+i} | m,n \in \mathbb{Z}\}, \\
\mathcal{G}'_{-\epsilon_i} &= span_{\mathbb{C}}\{\tilde{e}^*_{i}(m,n) = x^m y^n e_{N+i,0} - \overline{x^m y^n} e_{0,i} | m,n \in \mathbb{Z}\},
\end{aligned}
\tag{1.26}
$$

and

$$\mathcal{G}'_0 = span_{\mathbb{C}}\{\tilde{f}_{ii}(m,n) - \tilde{e}_0(m,n), \tilde{e}_0(p,s) | 1 \le i \le N, m,n \in \mathbb{Z}, p \notin \Lambda(q) \text{ or } s \notin \Lambda(q)\},$$

where $\tilde{e}_0(m,n) = (x^m y^n - \overline{x^m y^n}) e_{0,0}$.

Next we form a central extension of $\mathcal{G}'$

$$\widehat{\mathcal{G}}' = \mathcal{G}' \oplus \Big(\sum_{n \in \Lambda(q)} \oplus \mathbb{C}c(n) \Big) \oplus \mathbb{C}c_y \tag{1.27}$$

with Lie brackets as (1.7).

Remark 1.1 *Note that the index of the matrices in $M_{2N+1}(\mathbb{C}_q)$ ranges from 0 to $2N$.*

Now we have

Proposition 1.2

$$[\tilde{g}_{ij}(m,n), \tilde{g}_{kl}(p,s)] = 0 \tag{1.28}$$

$$[\tilde{g}_{ij}(m,n), \tilde{f}_{kl}(p,s)] = -\delta_{il} q^{ms} \tilde{g}_{kj}(m+p, n+s) + \delta_{jl} q^{(s-n)m} \tilde{g}_{ki}(m+p, s-n) \tag{1.29}$$

$$
\begin{aligned}
&[\tilde{g}_{ij}(m,n), \tilde{h}_{kl}(p,s)] \\
= \; & -\delta_{ik} q^{-n(m+p)} \tilde{f}_{jl}(m+p, s-n) + \delta_{jk} q^{np} \tilde{f}_{il}(m+p, n+s) \\
& +\delta_{il} q^{-(mn+np+ps)} \tilde{f}_{jk}(m+p, -(n+s)) - \delta_{jl} q^{(n-s)p} \tilde{f}_{ik}(m+p, n-s) \\
& +m q^{np} \delta_{jk} \delta_{il} \delta_{m+p,0} \delta_{\overline{n+s},\bar{0}} (c(n+s) + c(-n-s)) \\
& -m \delta_{ik} \delta_{jl} \delta_{m+p,0} \delta_{\overline{n-s},\bar{0}} (c(n-s) + c(s-n))
\end{aligned}
\tag{1.30}
$$

$$[\tilde{g}_{ij}(m,n),\tilde{e}_k(p,s)]=0 \tag{1.31}$$

$$[\tilde{g}_{ij}(m,n),\tilde{e}^*_k(p,s)]=-\delta_{ik}q^{-n(m+p)}\tilde{e}_j(m+p,s-n)+\delta_{jk}q^{np}\tilde{e}_i(m+p,n+s) \tag{1.32}$$

$$[\tilde{g}_{ij}(m,n),\tilde{e}_0(p,s)]=0 \tag{1.33}$$

$$\begin{aligned}[][\tilde{f}_{ij}(m,n),\tilde{f}_{kl}(p,s)] &= \delta_{jk}q^{np}\tilde{f}_{il}(m+p,n+s)-\delta_{il}q^{sm}\tilde{f}_{kj}(m+p,n+s)\\ &\quad +2mq^{np}\delta_{jk}\delta_{il}\delta_{m+p,0}\delta_{\overline{n+s},\overline{0}}c(n+s)\end{aligned} \tag{1.34}$$

$$[\tilde{f}_{ij}(m,n),\tilde{h}_{kl}(p,s)]=-\delta_{ik}q^{-n(m+p)}\tilde{h}_{jl}(m+p,s-n)-\delta_{il}q^{ms}\tilde{h}_{kj}(m+p,n+s) \tag{1.35}$$

$$[\tilde{f}_{ij}(m,n),\tilde{e}_k(p,s)]=\delta_{jk}q^{np}\tilde{e}_i(m+p,n+s) \tag{1.36}$$

$$[\tilde{f}_{ij}(m,n),\tilde{e}^*_k(p,s)]=-\delta_{ik}q^{-n(m+p)}\tilde{e}^*_j(m+p,s-n) \tag{1.37}$$

$$[\tilde{f}_{ij}(m,n),\tilde{e}_0(p,s)]=0 \tag{1.38}$$

$$[\tilde{h}_{ij}(m,n),\tilde{h}_{kl}(p,s)]=0 \tag{1.39}$$

$$[\tilde{h}_{ij}(m,n),\tilde{e}_k(p,s)]=\delta_{jk}q^{np}\tilde{e}^*_i(m+p,n+s)-\delta_{ik}q^{-n(m+p)}\tilde{e}^*_j(m+p,s-n) \tag{1.40}$$

$$[\tilde{h}_{ij}(m,n),\tilde{e}^*_k(p,s)]=0 \tag{1.41}$$

$$[\tilde{h}_{ij}(m,n), \tilde{e}_0(p,s)] = 0 \tag{1.42}$$

$$[\tilde{e}_i(m,n), \tilde{e}_k(p,s)] = q^{m(s-n)}\tilde{g}_{ki}(m+p, s-n) \tag{1.43}$$

$$\begin{aligned} & [\tilde{e}_i(m,n), \tilde{e}_k^*(p,s)] \\ = \; & -\delta_{ik}q^{-n(m+p)}\tilde{e}_0(m+p, s-n) - q^{p(n-s)}\tilde{f}_{ik}(m+p, n-s) \\ & +m\delta_{ik}\delta_{m+p,0}\delta_{\overline{n-s},\overline{0}}(c(n-s)+c(s-n)) \end{aligned} \tag{1.44}$$

$$[\tilde{e}_i(m,n), \tilde{e}_0(p,s)] = q^{np}\tilde{e}_i(m+p, n+s) - q^{p(n-s)}\tilde{e}_i(m+p, n-s) \tag{1.45}$$

$$[\tilde{e}_i^*(m,n), \tilde{e}_k^*(p,s)] = q^{m(s-n)}\tilde{h}_{ki}(m+p, s-n) \tag{1.46}$$

$$[\tilde{e}_i^*(m,n), \tilde{e}_0(p,s)] = q^{np}\tilde{e}_i^*(m+p, n+s) - q^{p(n-s)}\tilde{e}_i^*(m+p, n-s) \tag{1.47}$$

$$\begin{aligned} & [\tilde{e}_0(m,n), \tilde{e}_0(p,s)] \\ = \; & (q^{np} - q^{sm})\tilde{e}_0(m+p, n+s) + (q^{m(s-n)} - q^{-n(m+p)})\tilde{e}_0(m+p, s-n) \\ & +mq^{np}\delta_{m+p,0}\delta_{\overline{n+s},\overline{0}}(c(n+s)+c(-n-s)) \\ & -m\delta_{m+p,0}\delta_{\overline{n-s},\overline{0}}(c(n-s)+c(s-n)) \end{aligned} \tag{1.48}$$

for all $m, p, n, s \in \mathbb{Z}$ *and* $1 \le i, j, k, l \le N$.

The proof of Proposition 1.2 is similar to Proposition 1.1.

Remark 1.2 *Note that unlike (1.4), the anti-involution in [AABGP] is given by*

$$\bar{x} = \pm x, \quad \bar{y} = \pm y.$$

2 Representations

In this section, we follow the idea in [G] and [FF] to construct representations for the three types of BC_N-graded Lie algebras which are given in Section 1.

Let $\mathcal{R}$ be an associative algebra. Let $\rho = \pm 1$. We define a ρ-bracket on $\mathcal{R}$ as follow:

$$\{a, b\}_\rho = ab + \rho ba, \quad a, b \in \mathcal{R}. \tag{2.1}$$

It is easy to see that

$$\{a, b\}_\rho = \rho\{b, a\}_\rho \text{ and } [ab, c] = a\{b, c\}_\rho - \rho\{a, c\}_\rho b \tag{2.2}$$

for $a, b, c \in \mathcal{R}$, where $[a, b] = \{a, b\}_{-1}$ is the Lie bracket.

2.1 Type C and D

Define $\mathfrak{a}$ to be the unital associative algebra with $2N$ generators $a_i, a_i^*, 1 \le i \le N$, subject to relations

$$\{a_i, a_j\}_\rho = \{a_i^*, a_j^*\}_\rho = 0, \quad \text{and} \quad \{a_i, a_j^*\}_\rho = \rho\delta_{ij}. \tag{2.3}$$

Let the associative algebra $\alpha(N, \rho)$ be generated by

$$\{u(m) | u \in \bigoplus_{i=1}^{N} (\mathbb{C}a_i \oplus \mathbb{C}a_i^*), m \in \mathbb{Z}\} \tag{2.4}$$

with the relations

$$\{u(m), v(n)\}_\rho = \{u, v\}_\rho \delta_{m+n,0}. \tag{2.5}$$

We now define the normal ordering as in [FF](see also [F2]).

$$\begin{aligned} : u(m)v(n) : \quad &= \begin{cases} u(m)v(n) & \text{if } n > m, \\ \frac{1}{2}\big(u(m)v(n) - \rho v(n)u(m)\big) & \text{if } m = n, \\ -\rho v(n)u(m) & \text{if } m > n, \end{cases} \\ &= -\rho : v(n)u(m) : \end{aligned} \tag{2.6}$$

for $n, m \in \mathbb{Z}, u, v \in \mathfrak{a}$. Set

$$\theta(n) = \begin{cases} 1, & \text{for } n > 0, \\ \frac{1}{2}, & \text{for } n = 0, \\ 0, & \text{for } n < 0, \end{cases} \quad \text{then } 1 - \theta(n) = \theta(-n). \tag{2.7}$$

We have

$$
(2.8) \qquad \begin{aligned} :a_i(m)a_j(n): &= a_i(m)a_j(n) = -\rho a_j(n)a_i(m),\\ :a_i^*(m)a_j^*(n): &= a_i^*(m)a_j^*(n) = -\rho a_j^*(n)a_i^*(m). \end{aligned}
$$

and

$$
(2.9) \qquad \begin{aligned} a_i(m)a_j^*(n) &=: a_i(m)a_j^*(n): +\rho\delta_{ij}\delta_{m+n,0}\theta(m-n),\\ a_j^*(n)a_i(m) &= -\rho :a_i(m)a_j^*(n): +\delta_{ij}\delta_{m+n,0}\theta(n-m). \end{aligned}
$$

It follows from (2.2) that

$$
(2.10) \qquad \begin{aligned} &[a_i(m)a_j(n), a_k(p)] = 0,\\ &[a_i(m)a_j(n), a_k^*(p)] = -\delta_{ik}\delta_{m+p,0}a_j(n) + \rho\delta_{jk}\delta_{n+p,0}a_i(m),\\ &[a_i(m)a_j^*(n), a_k(p)] = \delta_{jk}\delta_{n+p,0}a_i(m),\\ &[a_i(m)a_j^*(n), a_k^*(p)] = -\delta_{ik}\delta_{m+p,0}a_j^*(n),\\ &[a_i^*(m)a_j^*(n), a_k(p)] = \delta_{jk}\delta_{n+p,0}a_i^*(m) - \rho\delta_{ik}\delta_{m+p,0}a_j^*(n),\\ &[a_i^*(m)a_j^*(n), a_k^*(p)] = 0, \end{aligned}
$$

for $m,n,p \in \mathbb{Z}, 1 \le i,j,k \le N$.

Let $\alpha(N,\rho)^+$ be the subalgebra generated by $a_i(n), a_j^*(m), a_k^*(0)$, for $n,m > 0$, and $1 \le i,j,k \le N$. Let $\alpha(N,\rho)^-$ be the subalgebra generated by $a_i(n)$, $a_j^*(m)$, $a_k(0)$, for $n,m < 0$, and $1 \le i,j,k \le N$. Those generators in $\alpha(N,\rho)^+$ are called annihilation operators while those in $\alpha(N,\rho)^-$ are called creation operators. Let $V(N,\rho)$ be a simple $\alpha(N,\rho)$-module containing an element v_0, called a“vacuum vector”, and satisfying

$$
(2.11) \qquad \alpha(N,\rho)^+ v_0 = 0.
$$

So all annihilation operators kill v_0 and

$$
(2.12) \qquad V(N,\rho) = \alpha(N,\rho)^- v_0.
$$

Now we are in the position to construct a class of fermions (if $\rho = 1$) or bosons (if $\rho = -1$) on $V(N,\rho)$. For any $m,n \in \mathbb{Z}, 1 \le i,j \le N$, set

$$
(2.13) \qquad f_{ij}(m,n) = \sum_{s\in\mathbb{Z}} q^{-ns} :a_i(m-s)a_j^*(s):,
$$

$$g_{ij}(m,n) = \sum_{s\in\mathbb{Z}} q^{-ns} : a_i(m-s)a_j(s) :, \tag{2.14}$$

$$h_{ij}(m,n) = \sum_{s\in\mathbb{Z}} q^{-ns} : a_i^*(m-s)a_j^*(s) : . \tag{2.15}$$

Although $f_{ij}(m,n), g_{ij}(m,n)$ and $h_{ij}(m,n)$ are infinite sums, they are well-defined as operators on $V(N,\rho)$. Indeed, for any vector $v \in V(N,\rho) = \alpha(N,\rho)^- v_0$, only finitely many terms in (2.13)-(2.15) can make a non-zero contribution to $g_{ij}(m,n)v$, $f_{ij}(m,n)v$ and $h_{ij}(m,n)v$.

Lemma 2.1 *We have*

$$\begin{aligned} g_{ij}(m,n) &= -\rho q^{-mn} g_{ji}(m,-n), \\ h_{ij}(m,n) &= -\rho q^{-mn} h_{ji}(m,-n). \end{aligned} \tag{2.16}$$

for m,n,p,s $\in \mathbb{Z}$ *and* $1 \le i,j,k,l \le N$.

Proof. We only prove for $g_{ij}(m,n)$. The proof of $h_{ij}(m,n)$ is similar.

$$\begin{aligned} g_{ij}(m,n) &= \sum_{s\in\mathbb{Z}} q^{-ns} : a_i(m-s)a_j(s) : \\ &= -\rho \sum_{s\in\mathbb{Z}} q^{-ns} : a_j(s)a_i(m-s) : \\ &= -\rho \sum_{s\in\mathbb{Z}} q^{-n(m-s)} : a_j(m-s)a_i(s) : \\ &= -\rho q^{-mn} g_{ji}(m,-n). \end{aligned}$$

∎

Lemma 2.2 *We have*

$$[g_{ij}(m,n), a_k(p)] = 0, \tag{2.17}$$

$$[g_{ij}(m,n), a_k^*(p)] = -\delta_{ik} q^{-n(m+p)} a_j(m+p) + \rho\delta_{jk} q^{np} a_i(m+p), \tag{2.18}$$

$$[g_{ij}(m,n), a_k(p)a_l(s)] = 0, \tag{2.19}$$

$$[g_{ij}(m,n), a_k(p)a_l^*(s)] = -\delta_{il}q^{-n(m+s)}a_k(p)a_j(m+s) + \rho\delta_{jl}q^{ns}a_k(p)a_i(m+s), \tag{2.20}$$

$$\begin{aligned}[g_{ij}(m,n), a_k^*(p)a_l^*(s)] = -\delta_{ik}q^{-n(m+p)}a_j(m+p)a_l^*(s) + \rho\delta_{jk}q^{np}a_i(m+p)a_l^*(s) \\ -\delta_{il}q^{-n(m+s)}a_k^*(p)a_j(m+s) + \rho\delta_{jl}q^{ns}a_k^*(p)a_i(m+s),\end{aligned} \tag{2.21}$$

$$[f_{ij}(m,n), a_k(p)] = \delta_{jk}q^{np}a_i(m+p), \tag{2.22}$$

$$[f_{ij}(m,n), a_k^*(p)] = -\delta_{ik}q^{-n(m+p)}a_j^*(m+p), \tag{2.23}$$

$$[f_{ij}(m,n), a_k(p)a_l^*(s)] = \delta_{jk}q^{np}a_i(m+p)a_l^*(s) - \delta_{il}q^{-n(m+s)}a_k(p)a_j^*(m+p), \tag{2.24}$$

$$[f_{ij}(m,n), a_k^*(p)a_l^*(s)] = -\delta_{ik}q^{-n(m+p)}a_j^*(m+p)a_l^*(s) - \delta_{il}q^{-n(m+s)}a_k^*(p)a_j^*(m+s), \tag{2.25}$$

$$[h_{ij}(m,n), a_k^*(p)] = 0, \tag{2.26}$$

$$[h_{ij}(m,n), a_k^*(p)a_l^*(s)] = 0, \tag{2.27}$$

for $m,n,p,s \in \mathbb{Z}$ *and* $1 \le i,j,k,l \le N$.

Proof. First, we have

$$\begin{aligned}
& [g_{ij}(m,n), a_k^*(p)] \\
= & \sum_{s\in\mathbb{Z}} q^{-ns}[: a_i(m-s)a_j(s) :, a_k^*(p)] \\
= & \sum_{s\in\mathbb{Z}} q^{-ns}[a_i(m-s)a_j(s), a_k^*(p)] \\
= & \sum_{s\in\mathbb{Z}} q^{-ns}\Big(a_i(m-s)\big\{a_j(s), a_k^*(p)\big\}_\rho - \rho\big\{a_i(m-s), a_k^*(p)\big\}_\rho a_j(s)\Big) \\
= & -\delta_{ik}q^{-n(m+p)}a_j(m+p) + \rho\delta_{jk}q^{np}a_i(m+p).
\end{aligned}$$

Then

$$\begin{aligned}
& [g_{ij}(m,n), a_k^*(p)a_l^*(s)] \\
= & [g_{ij}(m,n), a_k^*(p)]a_l^*(s) + a_k^*(p)[g_{ij}(m,n), a_l^*(s)] \\
= & -\delta_{ik}q^{-n(m+p)}a_j(m+p)a_l^*(s) + \rho\delta_{jk}q^{np}a_i(m+p)a_l^*(s) \\
& \quad -\delta_{il}q^{-n(m+s)}a_k^*(p)a_j(m+s) + \rho\delta_{jl}q^{ns}a_k^*(p)a_i(m+s).
\end{aligned}$$

So (2.18) and (2.21) hold true. The proof of the others is similar. ■

In what follows we shall mean $\frac{q^{mn}-1}{q^n-1} = m$ if $n \in \Lambda(q)$. This will make our formulas more concise.

Next we list all Lie brackets that are needed.

Proposition 2.1

$$[g_{ij}(m,n), g_{kl}(p,s)] = 0$$

for all $m,p,n,s \in \mathbb{Z}$ and $1 \le i,j,k,l \le N$.

Proposition 2.2

$$\begin{aligned}
& [g_{ij}(m,n), f_{kl}(p,s)] \\
= & -\delta_{il}q^{ms}g_{kj}(m+p,n+s) + \rho\delta_{jl}q^{(s-n)m}g_{ki}(m+p,s-n)
\end{aligned}$$

for all $m,p,n,s \in \mathbb{Z}$ and $1 \le i,j,k,l \le N$.

Proposition 2.3

$$\begin{aligned}
& [g_{ij}(m,n), h_{kl}(p,s)] \\
= & -\delta_{ik}q^{-n(m+p)}f_{jl}(m+p,s-n) + \rho\delta_{jk}q^{np}f_{il}(m+p,n+s) \\
& +\rho\delta_{il}q^{-(mn+np+ps)}f_{jk}(m+p,-(n+s)) - \delta_{jl}q^{(n-s)p}f_{ik}(m+p,n-s) \\
& -\rho\delta_{ik}\delta_{jl}\delta_{m+p,0}\frac{1}{2}(q^{s-n}+1)\frac{q^{m(s-n)}-1}{q^{s-n}-1} + \delta_{jk}\delta_{il}\delta_{m+p,0}q^{np}\frac{1}{2}(q^{s+n}+1)\frac{q^{m(s+n)}-1}{q^{s+n}-1}
\end{aligned}$$

for $m, p, n, s \in \mathbb{Z}$ *and* $1 \leq i, j, k, l \leq N$.

Proposition 2.4

$$\begin{aligned}[f_{ij}(m,n), f_{kl}(p,s)] &= \delta_{jk} q^{np} f_{il}(m+p, n+s) - \delta_{il} q^{sm} f_{kj}(m+p, n+s) \\ &\quad + \rho \delta_{jk} \delta_{il} q^{np} \delta_{m+p,0} \frac{1}{2}(q^{s+n}+1) \frac{q^{m(s+n)}-1}{q^{s+n}-1}\end{aligned}$$

for $m, p, n, s \in \mathbb{Z}$ *and* $1 \leq i, j, k, l \leq N$.

Proposition 2.5

$$[f_{ij}(m,n), h_{kl}(p,s)] = -\delta_{ik} q^{-n(m+p)} h_{jl}(m+p, s-n) - \delta_{il} q^{ms} h_{kj}(m+p, n+s)$$

for $m, p, n, s \in \mathbb{Z}$ *and* $1 \leq i, j, k, l \leq N$.

Proposition 2.6

$$[h_{ij}(m,n), h_{kl}(p,s)] = 0$$

for $m, p, n, s \in \mathbb{Z}$ *and* $1 \leq i, j, k, l \leq N$.

We shall only prove Proposition 2.3 which is the most complicated one. The proof of the others is either similar or easy.

Proof of Proposition 2.3 It follows from (2.21) and (2.7) that

$$\begin{aligned}
& [g_{ij}(m,n), q^{-st} : a_k^*(p-t) a_l^*(t) :] \\
= \; & -\delta_{ik} q^{-st-n(m+p-t)} a_j(m+p-t) a_l^*(t) + \rho \delta_{jk} q^{-st+n(p-t)} a_i(m+p-t) a_l^*(t) \\
& -\delta_{il} q^{-st-n(m+t)} a_k^*(p-t) a_j(m+t) + \rho \delta_{jl} q^{-st+nt} a_k^*(p-t) a_i(m+t) \\
= \; & -\delta_{ik} q^{-st-n(m+p-t)} \big(: a_j(m+p-t) a_l^*(t) : + \rho \delta_{jl} \delta_{m+p,0} \theta(m+p-2t)\big) \\
& + \rho \delta_{jk} q^{-st+n(p-t)} \big(: a_i(m+p-t) a_l^*(t) : + \rho \delta_{il} \delta_{m+p,0} \theta(m+p-2t)\big) \\
& -\delta_{il} q^{-st-n(m+t)} \big(-\rho : a_j(m+t) a_k^*(p-t) : + \delta_{jk} \delta_{m+p,0} \theta(p-m-2t)\big) \\
& + \rho \delta_{jl} q^{-st+nt} \big(-\rho : a_i(m+t) a_k^*(p-t) : + \delta_{ik} \delta_{m+p,0} \theta(p-m-2t)\big) \\
= \; & -\delta_{ik} q^{-n(m+p)} q^{-(s-n)t} : a_j(m+p-t) a_l^*(t) : \\
& + \rho \delta_{jk} q^{np} q^{-(n+s)t} : a_i(m+p-t) a_l^*(t) : \\
& + \rho \delta_{il} q^{-pn-ps-nm} q^{(s+n)(p-t)} : a_j(m+t) a_k^*(p-t) : \\
& -\delta_{jl} q^{p(n-s)} q^{-(n-s)(p-t)} : a_i(m+t) a_k^*(p-t) : \\
& -\rho \delta_{ik} \delta_{jl} \delta_{m+p,0} q^{-(s-n)t} (\theta(-2t) - \theta(-2m-2t)) \\
& + \delta_{jk} \delta_{il} \delta_{m+p,0} q^{np} q^{-(n+s)t} (\theta(-2t) - \theta(-2m-2t)).
\end{aligned}$$

Since

$$\begin{aligned}
&\sum_{t\in\mathbb{Z}} q^{-xt}\Big(\theta(-2t)-\theta(-2m-2t)\Big)\\
(2.28)\qquad &= \begin{cases} 0, & \text{if } m=0,\\ \frac{1}{2}\big(1+q^{xm}\big)+\sum_{t=-(m-1)}^{-1} q^{-xt}, & \text{if } m>0,\\ -\frac{1}{2}\big(1+q^{xm}\big)-\sum_{t=1}^{-m-1} q^{-xt}, & \text{if } m<0 \end{cases}\\
&= \frac{q^{(m+1)x}-q^{x}+q^{mx}-1}{2(q^{x}-1)}\\
&= \frac{1}{2}(q^{x}+1)\frac{q^{mx}-1}{q^{x}-1},
\end{aligned}$$

we obtain Proposition 2.3. ■

Next we shall find the correspondence between $g_{ij}(m,n)$, $h_{ij}(m,n)$, $f_{ij}(m,n)$ and $\tilde{g}_{ij}(m,n)$, $\tilde{h}_{ij}(m,n)$, $\tilde{f}_{ij}(m,n)$. To this end, we have to modify our operators $g_{ij}(m,n)$, $h_{ij}(m,n)$, $f_{ij}(m,n)$.

From Proposition 2.3, we see that , if $n+s\in\Lambda(q)$ and $n-s\in\Lambda(q)$,

$$\begin{aligned}
&[g_{ij}(m,n),h_{kl}(p,s)]\\
=\;& -\delta_{ik}q^{-n(m+p)}f_{jl}(m+p,s-n)+\rho\delta_{jk}q^{np}f_{il}(m+p,n+s)\\
&+\rho\delta_{il}q^{-(mn+np+ps)}f_{jk}(m+p,-(n+s))-\delta_{jl}q^{(n-s)p}f_{ik}(m+p,n-s)\\
&-\rho\delta_{ik}\delta_{jl}\delta_{m+p,0}m+\delta_{jk}\delta_{il}\delta_{m+p,0}q^{np}m.
\end{aligned}$$

If $n+s\in\mathbb{Z}\setminus\Lambda(q)$ and $n-s\in\Lambda(q)$,

$$\begin{aligned}
&[g_{ij}(m,n),h_{kl}(p,s)]\\
=\;& -\delta_{ik}q^{-n(m+p)}f_{jl}(m+p,s-n)+\rho\delta_{jk}q^{np}f_{il}(m+p,n+s)\\
&+\rho\delta_{il}q^{-(mn+np+ps)}f_{jk}(m+p,-(n+s))-\delta_{jl}q^{(n-s)p}f_{ik}(m+p,n-s)\\
&-\rho\delta_{ik}\delta_{jl}\delta_{m+p,0}m+\delta_{jk}\delta_{il}\delta_{m+p,0}q^{np}\frac{1}{2}(q^{s+n}+1)\frac{q^{m(s+n)}-1}{q^{s+n}-1}\\
=\;& -\delta_{ik}q^{-n(m+p)}f_{jl}(m+p,s-n)-\delta_{jl}q^{(n-s)p}f_{ik}(m+p,n-s)\\
&+\rho\delta_{jk}q^{np}\Big(f_{il}(m+p,n+s)-\frac{\rho}{2}\delta_{il}\delta_{m+p,0}\frac{q^{n+s}+1}{q^{n+s}-1}\Big)\\
&+\rho\delta_{il}q^{-(mn+np+ps)}\Big(f_{jk}(m+p,-n-s)-\frac{\rho}{2}\delta_{jk}\delta_{m+p,0}\frac{q^{-n-s}+1}{q^{-n-s}-1})\Big)\\
&-\rho\delta_{ik}\delta_{jl}\delta_{m+p,0}m.
\end{aligned}$$

Similarly, if $n+s\in\Lambda(q)$ and $n-s\in\mathbb{Z}\setminus\Lambda(q)$,

$$\begin{aligned}
&[g_{ij}(m,n),h_{kl}(p,s)]\\
=\;&\rho\delta_{jk}q^{np}f_{il}(m+p,n+s)+\rho\delta_{il}q^{-(mn+np+ps)}f_{jk}(m+p,-n-s)\\
&-\delta_{ik}q^{-n(m+p)}\Big(f_{jl}(m+p,s-n)-\frac{\rho}{2}\delta_{jl}\delta_{m+p,0}\frac{q^{s-n}+1}{q^{s-n}-1}\Big)\\
&-\delta_{jl}q^{(n-s)p}\Big(f_{ik}(m+p,n-s)-\frac{\rho}{2}\delta_{ik}\delta_{m+p,0}\frac{q^{n-s}+1}{q^{n-s}-1})\Big)\\
&+\delta_{jk}\delta_{il}\delta_{m+p,0}q^{np}m.
\end{aligned}$$

By the above two relations, we have if $n+s,n-s\in\mathbb{Z}\setminus\Lambda(q)$,

$$\begin{aligned}
&[g_{ij}(m,n),h_{kl}(p,s)]\\
=\;&\rho\delta_{jk}q^{np}f_{il}(m+p,n+s)+\rho\delta_{il}q^{-(mn+np+ps)}f_{jk}(m+p,-n-s)\\
&+\rho\delta_{jk}q^{np}\Big(f_{il}(m+p,n+s)-\frac{\rho}{2}\delta_{il}\delta_{m+p,0}\frac{q^{n+s}+1}{q^{n+s}-1}\Big)\\
&+\rho\delta_{il}q^{-(mn+np+ps)}\Big(f_{jk}(m+p,-n-s)-\frac{\rho}{2}\delta_{jk}\delta_{m+p,0}\frac{q^{-n-s}+1}{q^{-n-s}-1})\Big)\\
&-\delta_{ik}q^{-n(m+p)}\Big(f_{jl}(m+p,s-n)-\frac{\rho}{2}\delta_{jl}\delta_{m+p,0}\frac{q^{s-n}+1}{q^{s-n}-1}\Big)\\
&-\delta_{jl}q^{(n-s)p}\Big(f_{ik}(m+p,n-s)-\frac{\rho}{2}\delta_{ik}\delta_{m+p,0}\frac{q^{n-s}+1}{q^{n-s}-1})\Big).
\end{aligned}$$

Using the same method, from Proposition 2.4 we have, if $n+s\in\Lambda(q)$,

$$\begin{aligned}
&[f_{ij}(m,n),f_{kl}(p,s)]\\
=\;&\delta_{jk}q^{np}f_{il}(m+p,n+s)-\delta_{il}q^{sm}f_{kj}(m+p,n+s)+\rho\delta_{jk}\delta_{il}q^{np}\delta_{m+p,0}m.
\end{aligned}$$

If $n+s\in\mathbb{Z}\setminus\Lambda(q)$, then

$$\begin{aligned}
&[f_{ij}(m,n),f_{kl}(p,s)]\\
=\;&\delta_{jk}q^{np}\Big(f_{il}(m+p,n+s)-\frac{\rho}{2}\delta_{il}\delta_{m+p,0}\frac{q^{n+s}+1}{q^{n+s}-1}\Big)\\
&-\delta_{il}q^{sm}\Big(f_{kj}(m+p,n+s)-\frac{\rho}{2}\delta_{jk}\delta_{m+p,0}\frac{q^{n+s}+1}{q^{n+s}-1})\Big).
\end{aligned}$$

If we define

$$
F_{ij}(m,n)=\begin{cases} f_{ij}(m,n), & \text{for } n\in\Lambda(q)\\ f_{ij}(m,n)-\frac{1}{2}\rho\delta_{ij}\delta_{m,0}\frac{q^n+1}{q^n-1}, & \text{for } n\in\mathbb{Z}\setminus\Lambda(q)\end{cases}
$$

$$
G_{ij}(m,n)=g_{ij}(m,n),\quad H_{ij}(m,n)=h_{ij}(m,n), \tag{2.29}
$$

then we have

Theorem 2.1 *$V(N,\rho)$ is a module for the Lie algebra $\widehat{\mathcal{G}}_\rho$ under the action given by*

$$
\begin{aligned}
&\pi(\tilde{g}_{ij}(m,n))=G_{ij}(m,n), && \pi(\tilde{f}_{ij}(m,n))=F_{ij}(m,n),\\
&\pi(\tilde{h}_{ij}(m,n))=H_{ij}(m,n), && \pi(c(n))=\frac{\rho}{2}, \qquad \pi(c_y)=0.
\end{aligned}
$$

2.2 Type B

To consider BC_N-graded Lie algebras with grading subalgebra of type B_N, we require an extension of the algebra $\alpha(N,+)$. The generators

$$
\{e(m)|m\in\mathbb{Z}\} \tag{2.30}
$$

span an infinite-dimensional Clifford algebra with relations

$$
\{e(m),e(n)\}_+=e(m)e(n)+e(n)e(m)=\delta_{n+m,0}. \tag{2.31}
$$

Let $\alpha'(N)$ denote the algebra obtained by adjoining to $\alpha(N,+)$ the generators (2.30) with relations (2.31) and

$$
\{a_i(m),e(n)\}_+=0=\{a_i^*(m),e(n)\}_+ \tag{2.32}
$$

We now define the normal ordering as in (2.6), i.e.

$$
: e(m)e(n) :=\begin{cases} e(m)e(n) & \text{if } n>m\\ \frac{1}{2}\big(e(m)e(n)-e(n)e(m)\big) & \text{if } n=m\\ -e(n)e(m) & \text{if } n<m\end{cases},
$$

$$
\begin{aligned}
&: a_i(m)e(n) := a_i(m)e(n)=-e(n)a_i(m),\\
&: a_i^*(m)e(n) := a_i^*(m)e(n)=-e(n)a_i^*(m),
\end{aligned} \tag{2.33}
$$

for $n,m\in\mathbb{Z},1\le i,j\le N$. Then

$$
e(m)e(n) =: e(m)e(n) : +\delta_{m+n,0}\theta(m-n). \tag{2.34}
$$

By (2.2), we have

$$
(2.35)\qquad
\begin{aligned}
&[a_i(m)a_i(n), e(p)] = [a_i(m)a_i^*(n), e(p)] = [a_i^*(m)a_i^*(n), e(p)] = 0,\\
&[a_i(m)e(n), a_k(p)] = 0,\\
&[a_i(m)e(n), a_k^*(p)] = -\delta_{ik}\delta_{m+p,0}e(n),\\
&[a_i(m)e(n), e(p)] = \delta_{n+p,0}a_i(m),\\
&[a_i^*(m)e(n), a_k^*(p)] = 0,\\
&[a_i^*(m)e(n), e(p)] = \delta_{n+p,0}a_i^*(m).\\
&[e(m)e(n), e(p)] = \delta_{n+p,0}e(m) - \delta_{m+p,0}e(n).
\end{aligned}
$$

for $m, n, p \in \mathbb{Z}, 1 \leq i, j, k \leq N$.

Let V_0 be a simple Clifford module for the Clifford algebra generated by (2.30) with relations (2.31) and containing“vacuum vector” v_0', which is killed by annihilation operators. (Here we call $e(m)$ annihilation operator if $m > 0$, or a creation operator if $m < 0$. e(0) acts as scalar.)Because of (2.32), we see that the $\alpha'(N)$-module

$$(2.36)\qquad V'(N) = V(N, +) \otimes V_0 = \alpha'(N)v_0'$$

is simple.

Now we construct a class of fermions on $V'(N)$. For any $m, n \in \mathbb{Z}, 1 \leq i, j \leq N$, set

$$(2.37)\qquad f_{ij}(m, n) = \sum_{s\in\mathbb{Z}} q^{-ns} :a_i(m - s)a_j^*(s):,$$

$$(2.38)\qquad g_{ij}(m, n) = \sum_{s\in\mathbb{Z}} q^{-ns} :a_i(m - s)a_j(s):,$$

$$(2.39)\qquad h_{ij}(m, n) = \sum_{s\in\mathbb{Z}} q^{-ns} :a_i^*(m - s)a_j^*(s):,$$

$$(2.40)\qquad e_i(m, n) = \sum_{s\in\mathbb{Z}} q^{-ns} :a_i(m - s)e(s):,$$

$$(2.41)\qquad e_i^*(m, n) = \sum_{s\in\mathbb{Z}} q^{-ns} :a_i^*(m - s)e(s):,$$

$$e_0(m,n) = \sum_{s\in\mathbb{Z}} q^{-ns} : e(m-s)e(s) : . \tag{2.42}$$

Remark 2.1 *In this section, $g_{ij}(m,n), f_{ij}(m,n), h_{ij}(m,n)$ are the same as ones in the type D case (2.13)-(2.15) by taking $\rho = 1$. So we needn't to check the Lie brackets among them.*

Lemma 2.3 *We have*

$$[g_{ij}(m,n), a_k(p)e(s)] = [g_{ij}(m,n), e(p)e(s)] = 0, \tag{2.43}$$

$$[g_{ij}(m,n), a_k^*(p)e(s)] = -\delta_{ik} q^{-n(m+p)} a_j(m+p)e(s) + \delta_{jk} q^{np} a_i(m+p)e(s), \tag{2.44}$$

$$[f_{ij}(m,n), a_k(p)e(s)] = \delta_{jk} q^{np} a_i(m+p)e(s), \tag{2.45}$$

$$[f_{ij}(m,n), a_k^*(p)e(s)] = -\delta_{ik} q^{-n(m+p)} a_j^*(m+p)e(s), \tag{2.46}$$

$$[f_{ij}(m,n), e(p)e(s)] = 0, \tag{2.47}$$

$$[h_{ij}(m,n), a_k(p)e(s)] = \delta_{jk} q^{np} a_i^*(m+p)e(s) - \delta_{ik} q^{-n(m+p)} a_j^*(m+p)e(s), \tag{2.48}$$

$$[h_{ij}(m,n), a_k^*(p)e(s)] = [h_{ij}(m,n), e(p)e(s)] = 0, \tag{2.49}$$

$$[e_i(m,n), a_k(p)] = 0, \tag{2.50}$$

(2.51) $$[e_i(m,n), a_k^*(p)] = -\delta_{ik} q^{-n(m+p)} e(m+p),$$

(2.52) $$[e_i(m,n), e(p)] = q^{np} a_i(m+p),$$

(2.53) $$[e_i(m,n), a_k(p)e(s)] = q^{ns} a_k(p) a_i(m+s),$$

(2.54) $$[e_i(m,n), a_k^*(p)e(s)] = -\delta_{ik} q^{-n(m+p)} e(m+p)e(s) + q^{ns} a_k^*(p) a_i(m+s),$$

(2.55) $$[e_i(m,n), e(p)e(s)] = q^{np} a_i(m+p)e(s) + q^{ns} a_i(m+s)e(p),$$

(2.56) $$[e_i^*(m,n), a_k^*(p)] = 0,$$

(2.57) $$[e_i^*(m,n), e(p)] = q^{np} a_i^*(m+p),$$

(2.58) $$[e_i^*(m,n), a_k^*(p)e(s)] = q^{ns} a_k^*(p) a_i^*(m+s),$$

(2.59) $$[e_i^*(m,n), e(p)e(s)] = q^{np} a_i^*(m+p)e(s) + q^{ns} a_i^*(m+s)e(p),$$

(2.60) $$[e_0(m,n), e(p)] = (q^{np} - q^{-n(m+p)}) e(m+p),$$

(2.61)
$$[e_0(m,n), e(p)e(s)] = (q^{np} - q^{-n(m+p)}) e(m+p)e(s) + (q^{ns} - q^{-n(m+s)}) e(p)e(m+s),$$

for m,n,p,s $\in \mathbb{Z}$ *and* $1 \le i, j, k \le N$.

As in Section 2.1, we have Propositions 2.1-2.6 plus the following propositions.

Proposition 2.7

$$[g_{ij}(m,n), e_k(p,s)] = [g_{ij}(m,n), e_0(p,s)] = 0,$$
$$[g_{ij}(m,n), e_k^*(p,s)] = -\delta_{ik} q^{-n(m+p)} e_j(m+p, s-n) + \delta_{jk} q^{np} e_i(m+p, n+s)$$

for all $m, p, n, s \in \mathbb{Z}$ *and* $1 \leq i, j, k \leq N$.

Proposition 2.8

$$[f_{ij}(m,n), e_k(p,s)] = \delta_{jk} q^{np} e_i(m+p, n+s),$$
$$[f_{ij}(m,n), e_k^*(p,s)] = -\delta_{ik} q^{-n(m+p)} e_j^*(m+p, s-n),$$
$$[f_{ij}(m,n), e_0(p,s)] = 0$$

for all $m, p, n, s \in \mathbb{Z}$ *and* $1 \leq i, j, k \leq N$.

Proposition 2.9

$$[h_{ij}(m,n), e_k(p,s)] = \delta_{jk} q^{np} e_i^*(m+p, n+s) - \delta_{ik} q^{-n(m+p)} e_j^*(m+p, s-n),$$
$$[h_{ij}(m,n), e_k^*(p,s)] = [h_{ij}(m,n), e_0(p,s)] = 0$$

for all $m, p, n, s \in \mathbb{Z}$ *and* $1 \leq i, j, k \leq N$.

Proposition 2.10

$$\begin{aligned}
[e_i(m,n), e_k(p,s)] &= q^{m(s-n)} g_{ki}(m+p, s-n), \\
[e_i(m,n), e_k^*(p,s)] &= -\delta_{ik} q^{-n(m+p)} e_0(m+p, s-n) - q^{p(n-s)} f_{ik}(m+p, n-s) \\
&\qquad -\delta_{ik}\delta_{m+p,0} \frac{1}{2}(q^{s-n}+1) \frac{q^{m(s-n)}-1}{q^{s-n}-1}, \\
[e_i(m,n), e_0(p,s)] &= q^{np} e_i(m+p, n+s) - q^{p(n-s)} e_i(m+p, n-s)
\end{aligned}$$

for all $m, p, n, s \in \mathbb{Z}$ *and* $1 \leq i, k \leq N$.

Proposition 2.11

$$\begin{aligned}
[e_i^*(m,n), e_k^*(p,s)] &= q^{m(s-n)} h_{ki}(m+p, s-n), \\
[e_i^*(m,n), e_0(p,s)] &= q^{np} e_i^*(m+p, n+s) - q^{p(n-s)} e_i^*(m+p, n-s)
\end{aligned}$$

for all $m, p, n, s \in \mathbb{Z}$ *and* $1 \leq i, k \leq N$.

Proposition 2.12

$$\begin{aligned}
& [e_0(m,n), e_0(p,s)] \\
= \quad & (q^{np} - q^{sm}) e_0(m+p, n+s) + \delta_{m+p,0} q^{np} \frac{1}{2}(q^{n+s}+1) \frac{q^{m(n+s)}-1}{q^{n+s}-1} \\
& + (q^{m(s-n)} - q^{-n(m+p)}) e_0(m+p, s-n) - \delta_{m+p,0} \frac{1}{2}(q^{s-n}+1) \frac{q^{m(s-n)}-1}{q^{s-n}-1}
\end{aligned}$$

for all $m, p, n, s \in \mathbb{Z}$.

We only give proofs for Proposition 2.10 and Proposition 2.12. The proof for the others is either similar or easy.

Proof of Proposition 2.10 and Proposition 2.12.

First, it follows from (2.53)-(2.55), (2.34) and (2.7) that

$$\begin{aligned}
& [e_i(m,n), q^{-st} : a_k(p-t)e(t) :] = q^{nt-st} a_k(p-t) a_i(m+t) \\
= \quad & q^{m(s-n)} q^{-(s-n)(m+t)} a_k(p-t) a_i(m+t) \\
= \quad & q^{m(s-n)} q^{-(s-n)(m+t)} : a_k(p-t) a_i(m+t) :,
\end{aligned}$$

$$\begin{aligned}
& [e_i(m,n), q^{-st} : a_k^*(p-t)e(t) :] \\
= \quad & q^{-st} \Big(-\delta_{ik} q^{-n(m+p-t)} e(m+p-t)e(t) - q^{nt} a_k^*(p-t) a_i(m+t) \Big) \\
= \quad & -\delta_{ik} q^{-n(m+p)} q^{-(s-n)t} e(m+p-t)e(t) + q^{-(s-n)t} a_k^*(p-t) a_i(m+t) \\
= \quad & -\delta_{ik} q^{-n(m+p)} q^{-(s-n)t} \Big(: e(m+p-t)e(t) : + \delta_{m+p,0} \theta(m+p-2t) \Big) \\
& - q^{-(s-n)t} \big(: a_i(m+t) a_k^*(p-t) : - \delta_{ik} \delta_{m+p,0} \theta(p-m-2t) \big) \\
= \quad & -\delta_{ik} q^{-n(m+p)} q^{-(s-n)t} : e(m+p-t)e(t) : \\
& - q^{p(n-s)} q^{-(n-s)(p-t)} : a_i(m+t) a_k^*(p-t) : \\
& - \delta_{ik} \delta_{m+p,0} q^{-(s-n)t} \big(\theta(-2t) - \theta(-2m-2t) \big),
\end{aligned}$$

$$\begin{aligned}
& [e_i(m,n), q^{-st} : e(p-t)e(t) :] \\
= \quad & q^{-st} \big(q^{n(p-t)} a_i(m+p-t) e(t) + q^{nt} a_i(m+t) e(p-t) \big) \\
= \quad & q^{np} q^{-(n+s)t} : a_i(m+p-t) e(t) : + q^{p(n-s)} q^{-(n-s)(p-t)} : a_i(m+t) e(p-t) : .
\end{aligned}$$

Then by (2.28), we see that Proposition 2.10 holds true.

Secondly, it follows from (2.61), (2.34) and (2.28) that

$$\begin{aligned}
& [e_0(m,n), q^{-st} : e(p-t)e(t) :] \\
= \; & q^{-st}\Big((q^{n(p-t)} - q^{-n(m+p-t)})e(m+p-t)e(t) + (q^{nt} - q^{-n(m+t)})e(p-t)e(m+t)\Big) \\
= \; & q^{-st}(q^{n(p-t)} - q^{-n(m+p-t)})\big(: e(m+p-t)e(t) : + \delta_{m+p,0}\theta(m+p-2t)\big) \\
& + q^{-st}(q^{nt} - q^{-n(m+t)})\big(: e(p-t)e(m+t) : + \delta_{m+p,0}\theta(p-m-2t)\big) \\
= \; & q^{np}q^{-(s+n)t} : e(m+p-t)e(t) : - q^{-n(m+p)}q^{-(s-n)t} : e(m+p-t)e(t) : \\
& + q^{m(s-n)}q^{-(s-t)(m+t)} : e(p-t)e(m+t) : - q^{sm}q^{-(n+s)(m+t)} : e(p-t)e(m+t) : \\
& + \delta_{m+p,0}q^{np}q^{-(n+s)t}\big(\theta(-2t) - \theta(-2m-2t)\big) \\
& - \delta_{m+p,0}q^{-(s-n)t}\big(\theta(-2t) - \theta(-2m-2t)\big)
\end{aligned}$$

and Proposition 2.12 holds true. ∎

As in Section 2.1 of type D case, we need to modify the definition of our operators.

For Proposition 2.10, if $n - s \in \Lambda(q)$,

$$\begin{aligned}
& [e_i(m,n), e_k^*(p,s)] \\
= \; & -\delta_{ik}q^{-n(m+p)}e_0(m+p, s-n) - q^{p(n-s)}f_{ik}(m+p, n-s) - \delta_{ik}\delta_{m+p,0}m;
\end{aligned}$$

if $n - s \in \mathbb{Z} \setminus \Lambda(q)$,

$$\begin{aligned}
& [e_i(m,n), e_k^*(p,s)] \\
= \; & -\delta_{ik}q^{-n(m+p)}\Big(e_0(m+p, s-n) - \frac{1}{2}\delta_{m+p,0}\frac{q^{s-n}+1}{q^{s-n}-1}\Big) \\
& - q^{p(n-s)}\Big(f_{ik}(m+p, n-s) - \frac{1}{2}\delta_{jk}\delta_{m+p,0}\frac{q^{n-s}+1}{q^{n-s}-1})\Big).
\end{aligned}$$

For Proposition 2.12, if $n + s \in \Lambda(q)$ and $n - s \in \Lambda(q)$,

$$\begin{aligned}
& [e_0(m,n), e_0(p,s)] \\
= \; & (q^{np} - q^{sm})e_0(m+p, n+s) + (q^{m(s-n)} - q^{-n(m+p)})e_0(m+p, s-n) \\
& + \delta_{m+p,0}q^{np}m - \delta_{m+p,0}m;
\end{aligned}$$

if $n+s\in\mathbb{Z}\setminus\Lambda(q)$ and $n-s\in\Lambda(q)$,

$$\begin{aligned}&[e_0(m,n),e_0(p,s)]\\ =\;&(q^{np}-q^{sm})\Big(e_0(m+p,n+s)-\frac{1}{2}\delta_{m+p,0}\frac{q^{n+s}+1}{q^{n+s}-1}\Big)\\ &+(q^{m(s-n)}-q^{-n(m+p)})e_0(m+p,s-n)-\delta_{m+p,0}m;\end{aligned}$$

if $n+s\in\Lambda(q)$ and $n-s\in\mathbb{Z}\setminus\Lambda(q)$,

$$\begin{aligned}&[e_0(m,n),e_0(p,s)]\\ =\;&(q^{np}-q^{sm})e_0(m+p,n+s)+\delta_{m+p,0}q^{np}m\\ &+(q^{m(s-n)}-q^{-n(m+p)})\Big(e_0(m+p,s-n)-\frac{1}{2}\delta_{m+p,0}\frac{q^{s-n}+1}{q^{s-n}-1}\Big);\end{aligned}$$

if $n+s,n-s\in\mathbb{Z}\setminus\Lambda(q)$,

$$\begin{aligned}&[e_0(m,n),e_0(p,s)]\\ =\;&(q^{np}-q^{sm})\Big(e_0(m+p,n+s)-\frac{1}{2}\delta_{m+p,0}\frac{q^{n+s}+1}{q^{n+s}-1}\Big)\\ &+(q^{m(s-n)}-q^{-n(m+p)})\Big(e_0(m+p,s-n)-\frac{1}{2}\delta_{m+p,0}\frac{q^{s-n}+1}{q^{s-n}-1}\Big).\end{aligned}$$

Now we define

$$\begin{gathered}F_{ij}(m,n)=\begin{cases}f_{ij}(m,n), & \text{for } n\in\Lambda(q),\\ f_{ij}(m,n)-\frac{1}{2}\delta_{ij}\delta_{m,0}\frac{q^n+1}{q^n-1}, & \text{for } n\in\mathbb{Z}\setminus\Lambda(q),\end{cases}\\ G_{ij}(m,n)=g_{ij}(m,n),\quad H_{ij}(m,n)=h_{ij}(m,n),\\ E_i(m,n)=e_i(m,n),\quad E_i^*(m,n)=e_i^*(m,n),\\ E_0(m,n)=\begin{cases}e_0(m,n), & \text{for } n\in\Lambda(q),\\ e_0(m,n)-\frac{1}{2}\delta_{m,0}\frac{q^n+1}{q^n-1}, & \text{for } n\in\mathbb{Z}\setminus\Lambda(q).\end{cases}\end{gathered}\tag{2.62}$$

Then we have

Theorem 2.2 *$V'(N)$ is a module for the Lie algebra $\widehat{\mathcal{G}'}$ under the action given by*

$$\begin{aligned}\pi(\tilde{g}_{ij}(m,n))&=G_{ij}(m,n), & \pi(\tilde{f}_{ij}(m,n))&=F_{ij}(m,n),\\ \pi(\tilde{h}_{ij}(m,n))&=H_{ij}(m,n), & \pi(\tilde{e}_i(m,n))&=E_i(m,n),\\ \pi(\tilde{e}_i^*(m,n))&=E_i^*(m,n), & \pi(\tilde{e}_0(m,n))&=E_0(m,n),\\ \pi(c(n))&=\frac{1}{2}, & \pi(c_y)&=0.\end{aligned}$$

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